Classical diffusion and quantum level velocities: Systematic deviations from random matrix theory

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We study the response of the quasienergy levels in the context of quantized chaotic systems through the level velocity variance and relate them to classical diffusion coefficients using detailed semiclassical analysis. The systematic deviations from random matrix theory, assuming independence of eigenvectors from eigenvalues, are shown to be connected to classical higher-order time correlations of the chaotic system. We study the standard map as a specific example, and thus the well-known oscillatory behavior of the diffusion coefficient with respect to the parameter is reflected exactly in the oscillations of the variance of the level velocities. We study the case of mixed phase-space dynamics as well and note a transition in the scaling properties of the variance that occurs along with the classical transition to chaos. [S1063-651X(99)11810-5]

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I. INTRODUCTION

The quantum spectrum is well known to reflect in several ways classical integrability or its lack thereof [1,2,4]. For a completely chaotic, quantized system the energy eigenvalues have characteristic, in fact, universal fluctuation properties that coincide with random matrix theory (RMT) universality classes and the eigenfunction components are also distributed as Gaussian random variables. However, there are important deviations from this dull uniformity imposed by the underlying (asymptotic) deterministic chaos. Classical periodic orbits, a dense set of measure zero unstable orbits, introduce characteristic deviations that are well documented, including the phenomenon of eigenfunction scarring [3]. The movement of energy levels with the variation of an external parameter, level dynamics, has also been studied by several authors with different motivations [5–10]. It is known that the motion of the energy levels as a function of the parameter, now a psuedotime variable, is completely integrable whether the system is itself chaotic or not [11]. Nevertheless, there are characteristic features that are introduced by chaos, for instance, avoided crossings that may be characterized by the second derivative of the energy levels, i.e., the curvatures.

Here we study level ''velocities,'' and relate them directly to certain classical diffusion coefficients based on the diffusion of the variation of the action with an external parameter. Although we are using the term velocities, we are not discussing adiabatically changing a system, just the slopes of the level curves as a function of a controllable parameter. It has been known for some time that these are Gaussian distributed with a variance that has been related to a classical ''generalized conductance,'' especially in the context of weakly disordered metallic grains. Methods employed were mostly field theoretic and RMT-based, while numerical simulations of chaotic billiards led to the conjecture that the behavior of disordered systems could be extended to chaotic ones as well [8].

The variance has a significance beyond setting the scale of the Gaussian distribution of velocities. It enters as a normalization required to uncover possible universalities in parametric level correlations. It encodes the system specific characteristics of level motions as a function of an external parameter. Level correlations and velocities are experimentally accessible, for example, in microwave cavities [12] or quantum dots. Although universal parametric correlations are not well established experimentally, a recent experiment, exploiting the similarity of elastomechanical wave equations of flexural modes of plates to the Schrodinger equation, seems to lend support to it [13].

In the case when the changing parameters are Aharanov-Bohm flux lines that do not lead to any classical dynamical changes, but do lead to important spectral modifications, correlation between level velocities were semiclassically considered in [14]. For a treatment of Hamiltonian flows see [15]. Recent closely related work, in the context of Hamiltonian flows, is also found in [16] where detailed results about the variance of level velocities are presented for billiards.

We make precise the connection between classical diffusion and the variance of the level velocities in the simpler context of quantized maps or more generally time periodic systems where detailed semiclassical (and classical) analysis is possible. We evaluate the variance for the standard map as a function of the external kicking strength and show system specific correlations in the form of Bessel function oscillations. Since two-dimensional area preserving or more generally symplectic maps are Poincaré sections of Hamiltonian flows, our analysis also reflects upon these systems and is consistent with results derived therein. On the other hand, due to the vastly simpler numerical and analytical work involved with maps, they lend themselves to more detailed and extensive work.

We relate our analysis to a semiclassical evaluation of expectation values of generic operators in the eigenbasis, as well as touch upon two parameter variations and their correlations. The case when the dynamics leads to a mixed phase space is generic and we find a Weyl-type expansion in $\hbar$ for

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the variance. The principal contribution in this regime is well predicted by a simple classical correlation, which vanishes as the system undergoes a transition to chaos. The different scaling behaviors in effective $\hbar$ for mixed and chaotic systems can be experimentally observed. We will consider the standard map as an example. Others before us have used such systems to study level dynamics [9,17].

II. THE STANDARD MAP AND RANDOM MATRIX THEORY

Here we define the model studied below and derive the RMT predictions for these. Let the classical Hamiltonian have the form

$$H = p^2/2 - \lambda V(q) \sum_{n=-\infty}^{\infty} \delta(t-n), \qquad (1)$$

so that the Floquet operator connecting states just before kicks is given by

$$U = \exp(-ip^2/2\hbar)\exp[i\lambda V(q)/\hbar]. \qquad (2)$$

The time between kicks is taken to be unity, as there are two independent parameters already present, namely, λ and N. Such systems, known as quantum maps, were first studied in [18,19] and led to the uncovering of dynamical localization, akin to Anderson localization in disordered conductors [20]. We will typically consider the above to be the way the parameter of interest (λ) enters the problem.

While this is a map on the plane (for one-degree-of-freedom systems), we consider their restriction to the torus $[0,1)^2$. This is essential as we have in mind bounded Hamiltonian systems and not open scattering ones. Periodic boundary conditions are imposed in both p and q directions. We will assume that $V(q)$ is a smooth function on [0, 1) with unit periodicity. Denote its average as

$$\bar{V} = \int_0^1 V(q)dq. \qquad (3)$$

Let the quantum map be the N dimensional unitary matrix operator denoted by U. Maps, such as the standard map restricted to a torus, are quantized using standard canonical quantization [21]. Periodic boundary conditions in both canonical variables impose a finite number of states, which is the inverse effective Planck constant ($h = 1/N$). Thus, the classical limit is approached in the large N limit. Various quantum maps on the torus have been studied and form an important part of the literature on quantum chaos due to their inherent simplicity [22–25]. The discrete spectra (N levels) obtained are then analyzed for various properties, in particular, here the eigenangle velocities are obtained.

The classical standard map is given by the recursion

$$q_{i+1} = (q_i + p_{i+1}) \mod(1), \qquad (4)$$

$$p_{i+1} = [p_i - (k/2\pi)\sin(2\pi q_i)] \mod(1),$$

where i is the discrete time. This is the solution to the Hamiltonian equations of motion for the potential $V(q) = \cos(2\pi q)$ and the Hamiltonian in Eq. (1). The dynamical

variables are monitored just before the kicks, and $\lambda = k/(2\pi)^2$. The standard map is of central importance as many other maps are locally described by this and the potential may be considered to be the first term in the Fourier expansion of more general periodic potentials. The parameter k is of principal interest and it controls the degree of chaos in the map; a complete transition to ergodicity is attained above values of $k \approx 5$, while the last rotational Kolmogorov-Arnold-Moser (KAM) torus breaks around $k \approx .971$ [26].

The quantum map in the discrete position basis is given by [27]

$$\langle n|U|n'\rangle = \frac{1}{\sqrt{iN}} \exp[i\pi(n-n')^2/N]$$

$$\times \exp\left(i\frac{kN}{2\pi}\cos[2\pi(n+a)/N]\right). \qquad (5)$$

The parameter to be varied will be the "kicking strength" k, while phase a will be used to avoid exact quantum symmetries, and n, $n' = 0,...,N-1$. The eigenvalue problem of the unitary matrix is written as $U|\psi_j\rangle = \exp(-i\phi_j)|\psi_j\rangle$. The eigenangles ϕ_j are real and their variation with the parameter k (level velocities) are given simply by the matrix elements

$$\frac{d\phi_j}{dk} = \frac{N}{2\pi}\langle\psi_j|V|\psi_j\rangle = \frac{N}{2\pi}\langle\psi_j|\cos(2\pi q)|\psi_j\rangle. \qquad (6)$$

The 2π factor is the result of choosing k as the relevant parameter and not $k/2\pi$ and we retain this as this corresponds to the more conventional usage where the last KAM torus breaks when the parameter value is just under unity.

It is then clear that studying level velocities is equivalent to studying expectation values of operators in the eigenbasis. Thus, if we require $\langle\psi_i|A|\psi_i\rangle$ we would look at the modified unitary operator (assuming A is Hermitian)

$$U = U_0 \exp(-i\lambda A/\hbar), \qquad (7)$$

where U_0 is the quantum system under study. Then, the expectation values are simply the corresponding level velocities evaluated at $\lambda = 0$, multiplied by $\hbar$. If one may identify the classical canonical transformation generated by A, we could study a modified classical map as well. However, since $\lambda = 0$, it is the properties of the original classical map that will be relevant. Reference [28] already discussed the general problem of semiclassical evaluation of matrix elements and our following work may be viewed in this context as well.

From the Gaussian distribution of eigenfunctions for a quantum chaotic system we expect the level velocities be similarly distributed. We will concentrate on the variance of these velocities, namely, the sum:

$$\sigma^2(k,N) = \frac{1}{N}\sum_{j=1}^{N}\left(\frac{d\phi_j}{dk}\right)^2 - \left[\frac{1}{N}\sum_{j=1}^{N}\left(\frac{d\phi_j}{dk}\right)\right]^2. \qquad (8)$$

We will assume, as is the case with the standard map example, that the average vanishes, i.e., $\bar{V} = 0$. Later we will generalize to the case of a nonvanishing average, or expec-

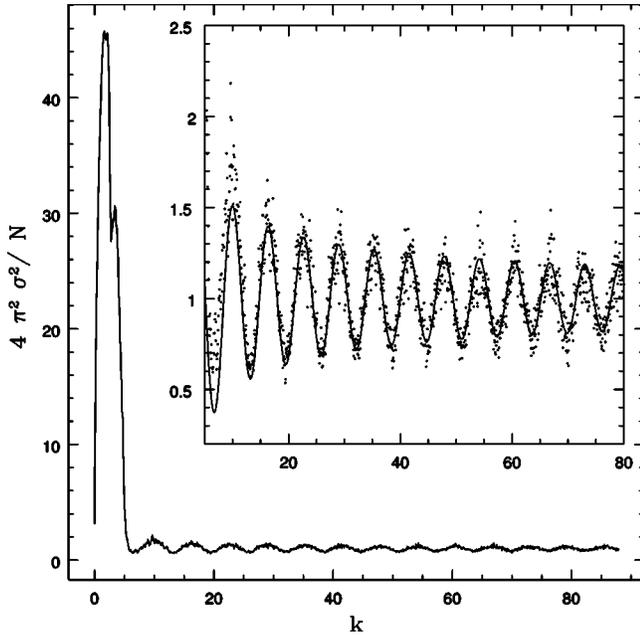FIG. 1. Scaled variance as a function of the parameter k, $N = 300$ and $a = 0.35$. The inset shows a part of the plot magnified; the points are numerical data while the smooth line is twice the diffusion coefficient. Quantities plotted are dimensionless.

tation values of operators with nonzero traces. Figure 1 shows a scaled σ^2 as a function of the parameter k. At about $k \approx 5$ the variance settles down to a near constant; this value coincides with the disappearance of major islands of stability in the classical phase space. What interests us primarily here, however, is the clear oscillations that persist as a function of k right into regions of large chaos as shown in the inset.

First, we study the value around which the oscillations occur, as this is provided by assuming RMT models. Using Eq. (6) we get

$$\sigma^2_{\text{RMT}} = \frac{N}{4\pi^2} \sum_{m=0}^{N-1} \left(\sum_{n=0}^{N-1} |\langle \phi_m|n\rangle|^2 V[(n+a)/N] \right)^2$$

$$= \frac{N}{4\pi^2} \sum_{m=0}^{N-1} \left(\sum_{n=0}^{N-1} |\langle \phi_m|n\rangle|^4 \{V[(n+a)/N]\}^2 \right.$$

$$+ \sum_{n \neq n'} |\langle \psi_m|n\rangle|^2 |\langle \phi_m|n'\rangle|^2 V[(n+a)/N]$$

$$\left. \times V[(n'+a)/N] \right). \tag{9}$$

The eigenfunctions have been expanded in a basis that diagonalizes the perturbation V, which we have taken to be the position basis. Since we assume a zero-centered or trace-less pertubation,

$$\sum_{n=0}^{N-1} V[(n+a)/N] = 0.$$

We use the square of this relation in Eq. (9) while replacing eigenfunction components by their ensemble averages (denoted by angular brackets) to derive that

$$\sigma^2_{\text{RMT}} = \frac{N^3}{4\pi^2} \overline{V^2} (\langle |\langle \psi_m|n\rangle|^4 \rangle - \langle |\langle \psi_m|n\rangle|^2 |\langle \psi_m|n'\rangle|^2 \rangle). \tag{10}$$

A crucial step in writing down the above is to assume the independence of the eigenfunction components from any specific position eigenvalues. While this is a reasonable statistical assumption we will see below that it misses important correlations that are incorporated naturally in semiclassical treatments. This is the origin of the nonuniversality of level dynamics, as this implies system-dependent correlation effects. The same perturbations (V) applied to different chaotic systems will result in different statistical responses, unlike the predictions of RMT.

We use standard results from RMT relevant to the Gaussian orthogonal ensemble, which is applicable here as well as the relevant circular ensembles, [29]. In particular,

$$\langle |\langle \psi_m|n\rangle|^4 \rangle = \frac{3}{N(N+2)} \sim \frac{3}{N^2}, \quad \langle |\langle \psi_m|n\rangle|^2 |\langle \psi_m|n'\rangle|^2 \rangle$$

$$= \frac{1}{N(N+2)} \sim \frac{1}{N^2}.$$

We finally get

$$\sigma^2_{\text{RMT}} = \frac{N}{2\pi^2} \overline{V^2}. \tag{11}$$

As a special case for the standard map $\overline{V^2} = \frac{1}{2}$ and we get $\sigma^2_{\text{RMT}} = N/4\pi^2$. This last result explains the value about which the oscillations occur in Fig. 1. This implies that the response of the system as measured by the movement of the energy levels is essentially the intensity of the perturbation. For chaotic systems then the response is independent of the system's detailed dynamical properties. We must also point out that when time reversal symmetry is broken the response is half as large. We now turn to the systematic oscillations that are not readily predicted by random matrix theory and are manifestly system dependent.

III. SEMICLASSICAL THEORY

A. The chaotic phase space

We first develop in some generality expressions for the variance of the level velocities in which semiclassical methods can be easily applied. We write a Gaussian smoothed density of states [30] as

$$\rho_M(\phi) = \sum_{n=-\infty}^{\infty} F_M(n) \exp(in\phi) \text{Tr } U^n, \tag{12}$$

where $F_M(n) = \exp(-n^2/2M^2)/(2\pi)$ is introduced to avoid divergences. The exact spiked density of states is obtained in the limit $M \to \infty$ although almost all levels will be resolved at $M = N$, as the mean level spacing is $2\pi/N$. The smoothed step function $N_M(\phi)$ is the integral of the level density with respect to ϕ. We derive then that

$$\int_0^{2\pi} \left(\frac{dN_M(\phi)}{dk} \right)^2 d\phi = \frac{MN\sigma^2(k,N)}{2\sqrt{\pi}}$$

$$= 2\pi \sum_{n=-\infty}^{\infty} \frac{F_M^2(n)}{n^2} \left| \frac{d}{dk} \text{Tr}(U^n) \right|^2. \tag{13}$$

The term $n=0$ does not belong in the sum, and it is understood that the first equality is an approximation that becomes exact as $M \to \infty$. From this expression it follows that it is the long time traces of the propagator, and therefore semiclassically, long periodic orbits that are important.

Another very similar route is through the identity

$$\text{Tr}(U^nV) = \sum_{j=1}^{N-1} \langle \psi_j|V|\psi_j \rangle \exp(-i\phi_j n), \tag{14}$$

thus implying that

$$\sigma^2(k,N) = \frac{N}{4\pi^2} \langle |\text{Tr}(U^nV)|^2 \rangle_n, \tag{15}$$

where the angular brackets indicate averaging over time n in the neighborhood of large n. We assume as is relevant for chaotic systems that there are no degeneracies. To make connections with the standard map above we would take $V = \cos(2\pi q)$.

Now we make use of the semiclassical approximation of the trace of the propagator as a sum over periodic orbits [1,31] that is

$$\text{Tr}(U^n) \sim n \sum_p A_p^{(n)} \exp(2\pi iN S_p^{(n)} - i\pi\nu_p/2), \tag{16}$$

where $A_p^{(n)} = 1/[2\sinh(\lambda_p n/2)]$, and the sum is over periodic orbits of period n that are labeled by p and have a Lyapunov exponent λ_p. The actions of these orbits are denoted by $S_p^{(n)}$ and are calculated from the generating function of the classical map. The phases ν_p are Maslov-like indices and will not concern us here.

There is also a generalization of the above, which is particularly easy to derive when the perturbation is diagonal in the position (or momentum) basis.

$$\text{Tr}(U^nV) \sim \sum_p A_p^{(n)} \exp(2\pi iN S_p^{(n)} - i\pi\nu_p/2) \sum_{j=1}^n V(x_j^p). \tag{17}$$

Here, $V(x_j^p)$ is the value of a phase-space representation of the operator V that is evaluated along the periodic orbit labeled p and at the point labeled j. An appropriate generalization in the energy domain for continuous time systems is found in [28]. The sum around the periodic orbit of the function V is essentially the derivative of the action with the parameter, and we may use either the first trace formula in conjunction with Eq. (13) or the second trace-formula with Eq. (15).

We take the second route as we connect with the first subsequently. Taking the modulus of the second trace formula gives

$$|\text{Tr}(U^nV)|^2 \sim \sum_p A_p^{(n)2} \left(\sum_{j=1}^n V(q_j^p) \right)^2 + \sum_{p \neq p'} A_p^{(n)} A_{p'}^{(n)}$$

$$\times \left(\sum_{j=1}^n V(q_j^p) \right) \left(\sum_{j=1}^n V(q_j^{p'}) \right)$$

$$\times \exp[2\pi iN(S_p - S_{p'})]. \tag{18}$$

As is usual, we have separated the diagonal contribution from the "off-diagonal," which corresponds to distinct pairs of orbits, with distinct actions. We have also assumed for simplicity, as is the case with the specific parameter variation chosen above in the standard map, that $V(x)$ is only position dependent; this does not alter the results below. We have also included the phases into the actions.

Since we expect that long periodic orbits are important, the diagonal approximation, which relies on random phases may be violated due to subtle correlations among their actions. The time at which we may expect action differences of the order of $\hbar$ is the so-called log-time, or Ehrenfest time. We argue that action differences are of the order of the orbit separation, and since areas of the order of $\hbar$ (for two-dimensional maps) would be populated with multiple periodic orbits beyond the log time, their action differences would also be comparable with $\hbar$. However, long periodic orbit actions are randomly distributed and will acquire correlations only around the Heisenberg time. At this time the off-diagonal terms will dominate the sum, as happens if we simply consider $\langle |\text{Tr}(U^n)|^2 \rangle_n$, which is asymptotically N, while the diagonal term is linearly increasing in time.

However, the off-diagonal term vanished due to the sums of $V(q)$ over very long periodic orbits. We may write for two distinct orbits p and p' after assuming uniform measure and replacing time averages over the periodic orbit, by the phase-space average that

$$\left(\sum_{j=1}^n V(q_j^p) \right) \left(\sum_{j=1}^n V(q_j^{p'}) \right) \sim n\bar{V}^2. \tag{19}$$

From our initial assumption that $\bar{V} = 0$, the off-diagonal term vanishes. The diagonal term is nonvanishing as we once again treat periodic orbits as ordinary long chaotic trajectories and derive that

$$\frac{1}{n} \left(\sum_{j=1}^n V(q_j^p) \right)^2 \equiv D(k) = C(0) + 2 \sum_{l=1}^n C(l), \tag{20}$$

where the time correlations are replaced by classical phase-space averages due to ergodicity.

$$C(l) = \langle V(q_0)V(q_l) \rangle = \frac{1}{\mathcal{A}} \int_{\mathcal{A}} dq_0 dp_0 V(q_0) V(f^l(q_0,p_0)), \tag{21}$$

where $f^l(q_0,p_0) = q_l$, f^l is the integrated dynamics in time l, and $\mathcal{A}$ denotes both the phase space and its area (in the cases considered this is unity). We assume that these exist, and are decreasing with l, typically exponentially for chaotic systems and that a few terms may be sufficient. This is not established in generality and complications may arise due to mar-

ginally stable orbits leading to nonexponential behaviors. For the standard map, coefficients up to $C(2)$ are dominant and sufficient to see the essential behavior. We have dropped the index p as now we will treat such long periodic orbits as generic nonperiodic orbits. Indeed by using the ergodic theorem we have already abandoned any particularities that may arise due to the orbit being periodic. Later, we remark on a case when we may not neglect off-diagonal terms.

The alternative route is to take the derivative of the first trace formula and use Eq. (13). Again we neglect the off-diagonal terms for reasons given above. We will assume that the derivatives of the actions with the parameter (action velocities), for a given period or period interval, are such that their average is zero while their variance is proportional to the time period. This assumption is equivalent to the vanishing of the phase-space average of $V(q)$ and the presence of ergodicity. This was noted for general Hamiltonian systems in [32], and we will see below in the context of maps how this simply arises. We replace then for each time n the individual action velocities (squared) by the variance,

$$\left\langle \left(\frac{dS_p^{(n)}}{dk} \right)^2 \right\rangle_p = D(k)n, \qquad (22)$$

where the angular brackets indicate the average over periodic orbits of period n.

In either approach, the uniformity principle [33] is applied in the form that there are e^{hn}/n orbits each with a Lyapunov exponent approximately λ per unit time. Then $|A_p^{(n)}|^2 \approx e^{-\lambda n}$, and assuming near equality of the topological entropy h and λ, we derive from Eq. (18) that

$$|\mathrm{Tr}(U^n V)|^2 \sim g D(k). \qquad (23)$$

Similarly from the other approach,

$$\frac{1}{n^2} \left| \frac{d}{dk} \mathrm{Tr}\, U^n \right|^2 \sim g \frac{N^2}{4\pi^2} D(k). \qquad (24)$$

The tilde sign in the above equations implies that the left-hand side can be expected to be the right-hand side in an average sense. The spread in time n will also reflect the spread in the average action velocity diffusion coefficient $D(k)$ with period. Results not shown here indicate that in the chaotic regime this is an exponential distribution. The factor g inserted above is due to the fact that symmetries can impose distinct orbits to have identical actions. This factor must be determined from classical and quantal symmetries, and includes phase-space symmetries. We finally get then from either approach the response in terms of the variance of the level velocities:

$$\sigma^2(k,N) = g \frac{N}{4\pi^2} D(k). \qquad (25)$$

Thus the variance of the level velocities is proportional to a classical diffusion coefficient that determines the diffusion of action velocities of periodic orbits. More explicit expressions for this coefficient are now derived. Area preserving maps such as the standard map have a generating function $L(q_{i+1}, q_i; k)$ from which the map may be derived as $\partial L/\partial q_i = -p_i$ and $\partial L/\partial q_{i+1} = p_{i+1}$ (Percival in [2]). The to-

tal action of a periodic orbit is equal to $S_p^{(n)} = \Sigma_i L(q_{i+1}, q_i; k)$, where the sum is over the n periodic points of the orbit p. Thus we derive, after assuming that the orbit is not at a point of bifurcation, that for a periodic orbit,

$$\frac{dS_p^{(n)}}{dk} = \frac{\partial S_p^{(n)}}{\partial k}. \qquad (26)$$

We have not used the partial derivative sign in defining the level velocities although we assume that only one parameter is varied. This is due to the subsequent fact that when the classical action derivative is written, it is a total derivative, in as much as the periodic orbit itself changes with the parameter. These two, however, are shown to be equal in the case of periodic orbits.

The variance is given by

$$\left\langle \left(\frac{dS_p^{(n)}}{dk} \right)^2 \right\rangle_p = \left\langle \sum_{i,j=1}^n \cos(2\pi q_i)\cos(2\pi q_j) \right\rangle_p$$

$$\sim \frac{n}{2}[1 + 2J_2(k)]. \qquad (27)$$

In the equality the sum is over different times along a given periodic orbit and then averaged over all periodic orbits of period n, while the approximation arises from a replacement of the average by the usual ensemble average and retaining up to the second-order time correlation. More precisely $C(0) = \frac{1}{2}$, $C(1) = 0$, and $C(2) = J_2(k)/2$, where $J_2(k)$ is a Bessel function. These are derived from

$$C(0) = \frac{1}{n} \sum_{i=1}^n \cos^2(2\pi q_i) \sim \int_0^1 \int_0^1 \cos^2(2\pi q)\, dq\, dp = \frac{1}{2},$$

$$C(1) = \frac{1}{n} \sum_{i=1}^n \cos(2\pi q_i)\cos(2\pi q_{i+1})$$

$$\sim \int_0^1 \int_0^1 \cos(2\pi q)\cos\{2\pi[q+p$$

$$- (k/2\pi)\sin(2\pi q)]\}\, dq\, dp = 0, \qquad (28)$$

$$C(2) = \frac{1}{n} \sum_{i=1}^n \cos(2\pi q_{i-1})\cos(2\pi q_{i+1}) \sim \int_0^1 \int_0^1$$

$$\times \cos[2\pi(q-p)]\cos\{2\pi[q+p-(k/2\pi)$$

$$\times \sin(2\pi q)]\}\, dq\, dp$$

$$= J_2(k)/2.$$

The symmetry factor is $g = 2$ for the standard map if we assume time-reversal invariance alone. This is the case for the data presented in Fig. 1 as we have intentionally broken the phase-space symmetry in the quantum system by assuming $a = 0.35$ (generic value) rather than $a = 0.5$, which will lead to twice the variance. Periodic orbits are either self-symmetric or more generically have symmetric partners with identical actions. The Bessel functions are characteristic of the diffusion coefficient in the standard map; the above simple derivation was proposed in the context of determin-

istic diffusion in [34]. However, we note that the diffusion relevant for level velocities is the diffusion in the action velocities and not the momentum, which is the usual quantity in which diffusion is studied [35].

The linear time dependence is a consequence of ergodicity, whereby time averages are replaced by phase-space averages, and the coefficient is also easy to find and is the scaling of the parameter introduced to uncover possible universalities in level dynamics. These then are the oscillations observed in the figure. The bold line in the inset is $2D(k) = [1+2J_2(k)]$. The significant deviation around the first minimum in the inset ($k \approx 6.5$) from theory could be due to the presence of small stable islands, which are the accelerator modes and are known to lead to anomalous transport in the standard map (Chirikov in [2]).

B. The mixed phase space

The regime where there is a mixed phase space consisting of large stable regions is generic, and in this case the analysis above fails: the assumptions about the trace formula and the uniformity principle operate only under conditions of complete hyperbolicity. While in the completely chaotic regime the variance scales as N, in the mixed phase-space regime it (principally) scales as N^2. This relates to the large hump in Fig. 1, to which we now turn our attention. One way of relating the variance to classical quantities is to recognize that

$$\sigma^2(k,N) = \frac{N}{4\pi^2} \langle \mathrm{Tr}(U^{-n} V U^n V) \rangle_n, \qquad (29)$$

where the average is once more the time average. Thus the variance of the quantum level velocities is directly the time average of operator autocorrelations. We consider the case when the average level velocity is zero, as the generalization is evident. Replacing operators by the corresponding classical observables, we expect to get the variance in the mixed phase regime. This is particularly successful as we are dealing with averages over the entire quantum spectrum. Thus we replace the trace operation divided by N with the classical phase-space average to get

$$\sigma^2(k,N)_{cl} = \frac{N^2}{4\pi^2} \int_0^1 dp_0 \int_0^1 dq_0 V(q_0) \langle V(q^{(n)}(q_0,p_0)) \rangle_n, \qquad (30)$$

where $q^{(n)}(q_0,p_0)$ is the position after n iterations starting from the initial condition (q_0,p_0). For the case in Fig. 1, $V(q) = \cos(2\pi q)$ and Fig. 2 compares in the mixed phase-space regime the exact quantum calculation with a purely classical simulation corresponding to $\sigma^2(k,N)_{cl}$. We see that a simple classical simulation reproduces the curve extremely well, including the secondary hump, until around $k \approx 2\pi$. It is quite remarkable that the classical curve continues to pick out the initial Bessel function oscillations in the deeply chaotic regime. In the figure the time average is done over an ensemble for the first hundred iterations, and the oscillations indicate short time correlations that will strictly disappear with increasing time.

The transition to classical chaos is accompanied then by a transition of the variance of the level velocities from a qua-
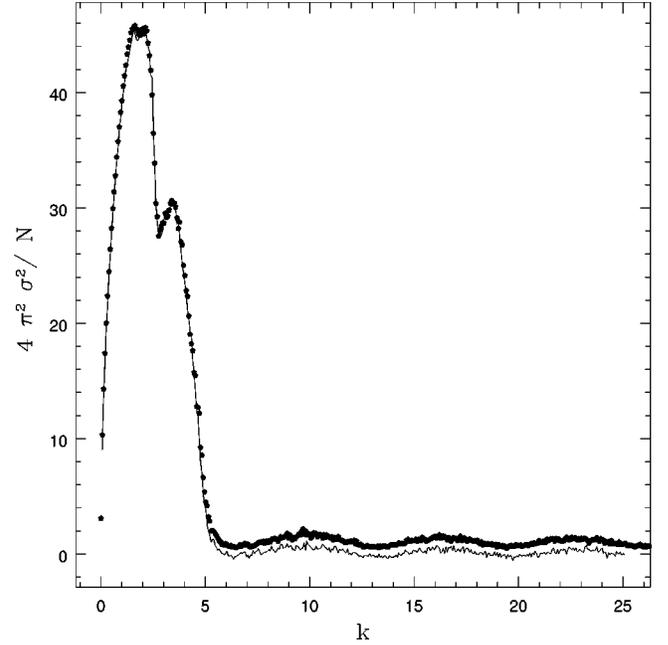

FIG. 2. Same as Fig. 1, comparing mainly in the mixed phase-space regime the exact variance (dots) with the classical estimate (line). The classical estimate is after averaging over a hundred time steps. Quantities plotted are dimensionless.

dratic to a linear N dependence. Based on this observation we may write a general expression for the variance of the level velocities as a Weyl series with principal terms

$$\sigma^2(k,N) = c_1(k)N + c_2(k)N^2, \qquad (31)$$

where $c_1(k)$ and $c_2(k)$ are system dependent and we have given above their expressions assuming only one of them is appreciable. Note that we have not evaluated $c_1(k)$ in the mixed phase-space regime, and that this will not in general vanish. On the other hand, we expect that $c_2(k)$ vanishes as a classical transition to chaos occurs. This is illustrated in Fig. 3, where $c_2(k)$ is evaluated based on a best fitting curve using five N values, equally spaced, between 100 and 500. The curve is fitted by assuming a third-order polynomial in N for which the coefficient of N^3 returned by the fit was always of the order of 10^{-6} or less.

In general, the RMT result derived earlier, Eq. (11), will be correct under the assumption that $C(l) = C(0)\delta_{0,l}$, implying delta correlated processes. Thus the departures from universality is related to higher-order time correlations. The response of the system is not only dependent on the strength of the perturbation, but also on the dynamical correlations inherent to the system.

C. Generalizations

We remark now on the general case $\bar{V} \neq 0$. This implies an overall drift to the energy levels due to changing phase-space volumes. Using Eq. (19) and adding and subtracting $n\bar{V}^2$ from the diagonal part of Eq. (18), we get after using $\langle |\mathrm{Tr}(U^n)|^2 \rangle_n = N$,

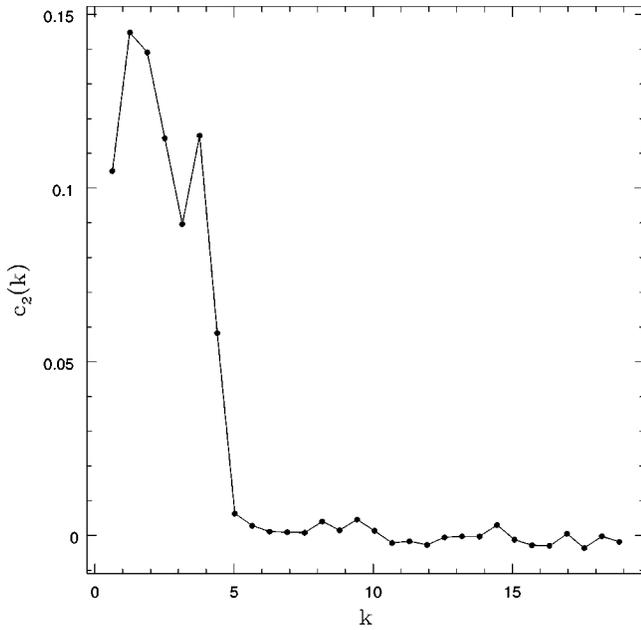$$\sigma^2(k,N) = \frac{N}{4\pi^2} g[D(k) - \bar{V}^2]. \qquad (32)$$

FIG. 3. The coefficient $c_2(k)$ as a function of the parameter; note that it reflects the classical transition to chaos. Quantities plotted are dimensionless.

The large chaos limit of this is

$$\sigma^2(\infty, N) = \frac{N}{4\pi^2} g(\overline{V^2} - \overline{V}^2), \qquad (33)$$

and is the RMT result.

Variations of two independent parameters is an important problem, considering that many novel effects, including geometric phases may be observed. Here we will consider, in a generalization of the above, correlations between independent parameter variations. We assume that the level velocities are given by the matrix elements

$$\frac{\partial \phi_j}{\partial \lambda_i} = \frac{N}{2\pi} \langle \psi_j | V_i | \psi_j \rangle, \qquad (34)$$

where $i = 1, 2$, and λ_i are two independent parameters while V_i are two Hermitian operators. Thus the correlations considered below are also correlations between diagonal matrix elements of two arbitrary Hermitian operators.

We derive then that

$$\frac{1}{N} \sum_{j=1}^{N} \frac{\partial \phi_j}{\partial \lambda_1} \frac{\partial \phi_j}{\partial \lambda_2} = \frac{N}{4\pi^2} \langle \text{Tr}(U^n V_1) \text{Tr}(U^{-n} V_2) \rangle_n. \qquad (35)$$

Using methods as outlined above the correlation function is semiclassically evaluated and we get

$$\sigma(\lambda_1, \lambda_2) \equiv \frac{1}{(\sigma_1 \sigma_2)} \left(\left\langle \frac{\partial \phi_j}{\partial \lambda_1} \frac{\partial \phi_j}{\partial \lambda_2} \right\rangle_j - \left\langle \frac{\partial \phi_j}{\partial \lambda_1} \right\rangle_j \left\langle \frac{\partial \phi_j}{\partial \lambda_2} \right\rangle_j \right)$$

$$\sim \frac{[D(\lambda_1, \lambda_2) - \overline{V_1}\, \overline{V_2}]}{\sqrt{(D_1 - \overline{V_1}^2)(D_2 - \overline{V_2}^2)}}. \qquad (36)$$
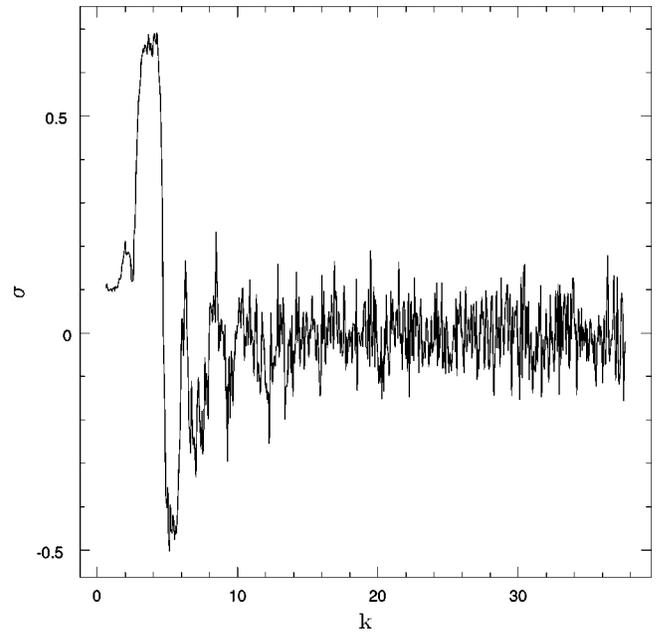

FIG. 4. The quantum correlation between the matrix elements of the two operators $\cos(2\pi q)$ and $\cos(4\pi q)$ whose classical correlation vanishes in the chaotic regime. Quantities plotted are dimensionless.

Function D is a generalization of Eq. (20) involving the dynamical correlation between the functions V_1 and V_2:

$$D(\lambda_1, \lambda_2) = \frac{1}{\mathcal{A}} \left(\sum_{l=-\infty}^{\infty} \int_{\mathcal{A}} V_1(q_0, p_0) V_2(q_l, p_l) dq_0 dp_0 \right); \qquad (37)$$

the dynamical variables after a time l integrated from (q_0, p_0) are denoted (q_l, p_l) while $D_{1,2}$ refer to the correlations appropriate to them individually and defined earlier in Eq. (20). Note that the backward and forward correlations $(l<0, l>0)$ are not, in general, equal. We remark that our derivations have assumed time-reversal symmetry, and that the factor g is responsible also for phase-space symmetries. It may be generalized to include the factors that come due to breakdown of time reversal, or inclusion of spin.

We finally consider the situation where not only the averages $\overline{V}_i$ vanish, but also there is no tangible correlation between them, i.e., $D(\lambda_1, \lambda_2) = 0$. Then, the semiclassical expressions above give zero and are incapable of capturing the small albeit finite and rapid oscillations (with parameter). This limitation is, of course, evident all along, including Fig. 1, where the Bessel function captures the low frequency oscillations only.

In Fig. 4, is one such example where we have considered $V_1 = \cos(2\pi q)$ and $V_2 = \cos(4\pi q)$. The parameter λ_2 is set at zero so that the relevant classical system is still the standard map, while the other parameter is k above. The correlation is seen as a function of this last parameter, the average actually vanishing. We see that this measure too captures the transition from mixed phase space to chaotic phase space, but that after the transition there are only extremely rapid oscillations about zero, although there are quite frequently fairly large correlations. In fact, the frequency of the oscillations are so rapid that they seem to have self-similar properties as a ran-

dom fractal. We may estimate the order of magnitude of the frequency if we assume that these arise from the off-diagonal part of the semiclassical sums. The magnitude of the parameter change needed to change a typical orbit action by h or $(1/N)$ is needed. From the fact that action changes have a variance proportional to the period, we get $|\Delta S| \sim \sqrt{n}|\Delta\lambda|$, and therefore $|\Delta\lambda| \sim N^{-3/2}$, if we take as the period $n = N$, which is the Heisenberg time and represents the time by which the spectrum is practically resolved.

In conclusion then, we have studied variances of level velocities and their generalizations in the chaotic as well as mixed phase-space regimes. Noting that the transition to chaos is perfectly reflected in this measure, we derived detailed formulas for them in terms of classical correlation coefficients and illustrated this with the help of the standard map. The mixed phase-space regime was surprisingly well captured by a simple classical estimate. The observations of oscillations or variations in the level velocity variances due to classical correlations, as well as using them to distinguish mixed from chaotic phase space, are both experimentally accessible.

The possibility of using the level velocity in conjunction with wave-function intensities in a measure of phase-space localization has been proposed [36], and it is hoped that this detailed understanding of the level velocities will help in this as well. In particular, this measure is a special case of the correlation between two operators discussed above, with an important complication being that the Wigner transform of the relevant operator, a projector in phase space, varies over scales of order $\hbar$. We have also noted that the RMT results, after assuming independence of eigenvalues and eignefunctions are capable of predicting the level velocities only in the limit of extremely large chaos, or equivalently ignoring all higher-order time correlations other than the zeroth.

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